

On the fully developed flow of a dense particulate mixture in a pipe

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Abstract

In this paper we present the governing equations for the flow of a dense particulate mixture in a pipe. The governing equations for the individual constituent as well as the mixture are provided based on continuum mechanics. Constitutive relations for the stress tensors and the interaction force are presented and discussed. A model is provided for the viscosity of the mixture. Numerical solution for the steady fully developed isothermal flow of such a mixture in a pipe is presented. The importance of the pressure drop and the skin friction in pipe flow are also discussed. © 1999 Elsevier Science S.A. All rights reserved.

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1. Introduction

Multiphase flows have increasingly become the subject of considerable attention [1] because of their importance in many industrial applications such as fluidized beds [2], pneumatic transport of solids [3], hydraulic transport [4], coal combustion [5], etc. For example, the flow behaviour of gas–particle mixtures in transport lines has been of interest in chemical processes for many years. In general, empirical correlations that predict the flow and pressure drop in such processes have been developed for specific ranges of solids and gas properties as well as for various geometries. Also, a great deal of research has been devoted to the use of coal-based slurries as retrofit fuels. Historically, coal–oil mixtures have received the most attention; their combustion characteristics and the flow or handling properties of these fuels have formed the basis of many research programs. A great deal of research emphasis has been put in the field of fluidization, generally applicable to any particulate material in the size range between 10 μm (diameter) and 3 mm.

Another area of interest involving flow of particles is the structural design of bins and silos. In processing minerals, in many processes such as the transport and combustion of coal, and in preventing or protecting against natural phenomena, such as avalanches or debris flows, it is essential to understand the factors governing the packing

and flow of powders and particulate matters [6]. Flowing granular materials can represent a limiting case of two-phase flow at high solids concentration and high solid-to-fluid density ratios. Many situations, such as discharge through bin outlets, flow through hoppers and chutes, pneumatic transport of coal, and fluidized beds require information or material properties of the granular media, flow patterns, concentration profile, etc. [7–9].

Much work has been reported in the literature about the mathematical modelling of multicomponent systems. The two foremost approaches use either volume-averaging technique or mixture theory. In the first approach [10–12], point-wise equations of motion, valid for a single fluid or a single particle, are modified to account for the presence of the other components and the interactions between components. These equations are then averaged over time or some suitable volume, which is large compared with some characteristic dimension (for example, particle spacing or the diameter of solid particle) but small compared to the dimensions of the whole system. The averaging process yields several terms, which represent the interactions between the constituents. Constitutive relations to represent these terms, as well as the stress tensor for each constituent are then required. Soo [1,13] provides a historical background with a detailed analysis for various multiphase flows in pipes, channels, etc. Marcus et al. [3] provide a thorough analysis of pneumatic transport of solid particles.

The second approach used in the mathematical modelling of multicomponent systems is mixture theory or the theory of interacting continua. This theory traces its root to

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the work of Fick [14] and was first given a rigorous mathematical structure by Truesdell [15]. It is a means for studying the interaction between several constituents by generalizing the equations and principles of the mechanics of a single continuum. The fundamental assumption in this theory is that at any instant of time, every point in space is occupied by one particle from each constituent. That is, each constituent is assumed to be homogenized to occupy the volume of the mixture. The historical development and the details of this theory are given in the works of Bowen [16], Atkin and Craine [17,18], Bedford and Drumheller [19] and Truesdell [20]. An updated and thorough analysis of mechanics of mixtures is given by Rajagopal and Tao [21].

In this paper we use the mixture theory approach and present the governing equations for the flow of a dense particulate mixture in a pipe. The equations for the individual constituent as well as the mixture are provided. Constitutive relations for the stress tensors and the interaction force are presented and discussed. A model is provided for the viscosity of the mixture based on the constitutive representation of the fluid and solid components. Numerical solution for the steady fully developed, isothermal, flow of such a mixture in a pipe is presented. The importance of the pressure drop and the skin friction in pipe flow are also discussed.

2. Governing equations

Since we are treating the mixture of a fluid and solid particles as a continuum, the balance laws valid for a single continuum would also apply here. Furthermore, inherent to the theory of interacting continua are the following ideas, attributed to Truesdell [20]:

- (i) All properties of the mixture must be mathematical consequences of properties of the constituents.
- (ii) So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.
- (iii) The motion of the mixture is governed by the same equations as is a single body.

With these, we proceed to present the governing equations. The conservation of mass for each constituent is given by the following.

2.1. Equations of motion for each constituent

$$\frac{\partial \rho_1}{\partial t} + \text{div}(\rho_1 \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial \rho_2}{\partial t} + \text{div}(\rho_2 \mathbf{v}) = 0, \quad (2)$$

where \mathbf{u} is the velocity of the solid particles, \mathbf{v} is the velocity of the fluid, $\partial/\partial t$ denotes the partial derivative with respect to time, and ρ_1 and ρ_2 represent the density of the solid and the fluid in the mixture, respectively. In order to show the relationship between these densities in the current configuration and their values in the reference configuration, we use the idea of volume fraction. Let:

$$\nu = \frac{\rho_1}{\rho_{10}}; \quad \phi = \frac{\rho_2}{\rho_{20}} = \frac{\rho_2}{\rho_f} \quad (3)$$

where ρ_1 denotes the density of solid particles in the current state in the mixture, and ρ_{10} is the density of the granular solid. Also, ρ_2 and ρ_{20} denote the density of the fluid in the current and reference state, respectively, by the volume additivity constraint [22]. If for simplicity we assign ρ_s and ρ_f as the densities of the solid particles and fluid in their reference states, respectively, and in addition, we assume the solid particles and the fluid are incompressible in the reference state, then Eqs. (1) and (2) can be rewritten as:

$$\frac{\partial \nu}{\partial t} + \text{div}(\nu \mathbf{u}) = 0 \quad (4)$$

$$\frac{\partial(1-\nu)}{\partial t} + \text{div}[(1-\nu)\mathbf{v}] = 0. \quad (5)$$

It is assumed that there is no interconversion of mass between the two components and that for a saturated mixture $\phi = 1 - \nu$.

The equations of the conservation of linear momentum take the form:

$$\rho_1 \frac{d\mathbf{u}}{dt} = \text{div} \mathbf{T}_1 - \mathbf{f} + \rho_1 \mathbf{b}, \quad (6)$$

$$\rho_2 \frac{d\mathbf{v}}{dt} = \text{div} \mathbf{T}_2 + \mathbf{f} + \rho_2 \mathbf{b}, \quad (7)$$

where d/dt is the material time derivative, \mathbf{T}_1 and \mathbf{T}_2 denote the partial stress tensors of the solid particles and the fluid, respectively, \mathbf{b} represents the body force, and \mathbf{f} represents the (mechanical) interaction between the components.

Furthermore, as we will discuss later, \mathbf{T}_1 and \mathbf{T}_2 are related to \mathbf{T}_s and \mathbf{T}_f in the following manner:

$$\mathbf{T}_1 = \mathbf{T}_s; \text{ and } \mathbf{T}_2 = (1 - \nu)\mathbf{T}_f. \quad (8)$$

The conservation of angular momentum is not considered here as it only implies (in this case) that the total stress tensor for the mixture is symmetric, though the partial stresses need not be symmetric. The conservation of energy is not considered here since we are interested in a purely mechanical system such as flow in a pipe, where any thermal or chemical effects are ignored.

2.2. Equations of motion for the mixture

To obtain the conservation of mass for the mixture, we add Eqs. (1) and (2). Therefore:

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \text{div}(\rho_1 \mathbf{u} + \rho_2 \mathbf{v}) = 0. \quad (9)$$

The mass-weighted velocity of the mixture, \mathbf{w} , is defined [16]:

$$\rho \mathbf{w} = \rho_1 \mathbf{u} + \rho_2 \mathbf{v}. \quad (10)$$

Thus, substituting Eq. (10) and the fact that $\rho = \rho_1 + \rho_2$ into Eq. (9) we obtain the conservation of mass for the mixture:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{w}) = 0. \quad (11)$$

To obtain the conservation of linear momentum for the mixture, we add Eqs. (6) and (7).

$$\rho_1 \frac{d\mathbf{u}}{dt} + \rho_2 \frac{d\mathbf{v}}{dt} = \text{div} \mathbf{T} + \rho \mathbf{b}, \quad (12)$$

where $\mathbf{T} = (\mathbf{T}_1 + \mathbf{T}_2)$ is the stress tensor for the mixture. Notice that we do not “force” the equation of linear momentum for the mixture, Eq. (12), to look like the following equation:

$$\rho \frac{d\mathbf{w}}{dt} = \text{div} \mathbf{T}^* + \rho \mathbf{b}, \quad (13)$$

where now the stress tensor of the mixture \mathbf{T}^* is no longer defined as the sum of the partial stresses. Both approaches have been used in mixture theory. Green and Naghdi [23] provide an interesting explanation as to why the form of the equation of linear momentum, given by Eq. (12) may be more appropriate than the form given by Eq. (13).

2.3. Constitutive relations for the stress and the interactive force

Based on our knowledge of modeling in the theory of granular materials and linearly viscous fluid [24,25], it would be natural to assume that all the constitutive functions, i.e., \mathbf{T}^s , \mathbf{T}^f , and \mathbf{f} depend on:

$$\rho_1, \rho_2, \nabla \rho_1, \nabla \rho_2, \nabla \nabla \rho_1, \nabla \nabla \rho_2, \mathbf{u}^f - \mathbf{u}^s, \mathbf{D}^s, \mathbf{D}^f, \dots \quad (14)$$

Where \mathbf{D}^s and \mathbf{D}^f are the symmetric parts of the velocity gradients of the solid particles and the fluid, respectively, defined in Eq. (25), and ∇ is the gradient operator. The interaction \mathbf{f} , in addition could depend on a frame-indifferent quantity derivable from the acceleration \mathbf{a}^s and \mathbf{a}^f of the solid and the fluid acceleration, respectively, to account for the virtual mass effect. Then, using methods which are by now standard in continuum mechanics [17,26–29], we can obtain restrictions on the forms appropriate for such constitutive expressions. Here, we shall not discuss such an approach. Another approach to modeling the mixture is to postulate the constitutive expressions by simply generalizing the structure of the constitutive relations using the continuum theory. This is what we shall do here. In this case:

$$\mathbf{T}^s = \mathbf{T}^s(\rho_1, \rho_2, \nabla \rho_1, \nabla \rho_2, \nabla \nabla \rho_1, \nabla \nabla \rho_2, \mathbf{u}^f - \mathbf{u}^s, \mathbf{D}^s, \mathbf{D}^f) \quad (15)$$

$$\mathbf{T}^f = \mathbf{T}^f(\rho_1, \rho_2, \nabla \rho_1, \nabla \rho_2, \nabla \nabla \rho_1, \nabla \nabla \rho_2, \mathbf{u}^f - \mathbf{u}^s, \mathbf{D}^s, \mathbf{D}^f). \quad (16)$$

In the case of a single constituent granular solid, the Cauchy stress \mathbf{T}^s would depend on the arguments that have the superscript s associated with them. Such a model would be a generalization of the model derived by Goodman and Cowin [30,31] to describe the mechanics of granular materials. Due to the presence of the term $\nabla \nabla \rho_1$, such a theory would include the model proposed by Kortweg to describe the effects of capillarity [32]. It is also worth observing that such models exhibit the phenomena of normal stress differences [24].

In general, the constitutive expressions for \mathbf{T}^s and \mathbf{T}^f in Eqs. (15) and (16) would depend on the kinematical quantities associated with both constituents. However, we shall assume that \mathbf{T}^s and \mathbf{T}^f depend only on the kinematical quantities associated with the solid and fluid, respectively. This assumption is sometimes called the “principle of phase separation” [33]. This is not a principle but just an assumption [34,35]. For the sake of simplicity, we shall assume that \mathbf{T}^s and \mathbf{T}^f do not depend on $\nabla \nabla \rho_1$ and $\nabla \nabla \rho_2$, respectively. By using the standard methods in modern continuum mechanics, i.e., the principle of material frame indifference, an isotropic representation of Eq. (15) is [32]:

$$\begin{aligned} \mathbf{T}^s = & \mathbf{a}_0 \mathbf{1} + \mathbf{a}_1 \mathbf{D}^s + \mathbf{a}_2 \mathbf{N} + \mathbf{a}_3 \mathbf{D}^{s^2} + \mathbf{a}_4 \mathbf{N}^2 \\ & + \mathbf{a}_5 (\mathbf{D}^s \mathbf{N} + \mathbf{N} \mathbf{D}^s) + \mathbf{a}_6 (\mathbf{D}^{s^2} \mathbf{N} + \mathbf{N} \mathbf{D}^{s^2}) \\ & + \mathbf{a}_7 (\mathbf{D}^s \mathbf{N}^2 + \mathbf{N}^2 \mathbf{D}^s) + \mathbf{a}_8 (\mathbf{D}^{s^2} \mathbf{N}^2 + \mathbf{N}^2 \mathbf{D}^{s^2}), \end{aligned} \quad (17)$$

where:

$$\mathbf{N} = \nabla \rho_1 \otimes \nabla \rho_1, \quad (18)$$

where \otimes denotes the outer product of two vectors and the coefficients \mathbf{a}_i , $i = 0-8$, in general, depend on ρ_1 , and the invariants $\text{tr } \mathbf{D}^s$, $\text{tr } \mathbf{D}^{s^2}$, $\text{tr } \mathbf{D}^{s^3}$, $\text{tr } \mathbf{N}$, $\text{tr } \mathbf{N}^2$, $\text{tr } \mathbf{N}^3$, $\text{tr } (\mathbf{D}^s \mathbf{N})$, $\text{tr } (\mathbf{D}^s \mathbf{N}^2)$, $\text{tr } (\mathbf{N} \mathbf{D}^{s^2})$, $\text{tr } (\mathbf{N}^2 \mathbf{D}^{s^2})$. If, in addition, we require that \mathbf{T}^s depends linearly on \mathbf{D}^s and \mathbf{N} , this model simplifies considerably:

$$\begin{aligned} \mathbf{T}^s = & \{ \beta_0(\rho_1) + \beta_1(\rho_1) \nabla \rho_1 \cdot \nabla \rho_1 + \beta_2(\rho_1) \text{tr } \mathbf{D}^s \} \mathbf{1} \\ & + \beta_4(\rho_1) \nabla \rho_1 \otimes \nabla \rho_1 + \beta_3(\rho_1) \mathbf{D}^s. \end{aligned} \quad (19)$$

The details of this derivation and the physical meanings of the phenomenological coefficients are given in the works of Rajagopal and Massoudi [24] and Rajagopal et al. [36]. A similar model to this was also proposed by Savage [37]. The spherical part of the stress tensor in Eq. (19) can be interpreted as the solid pressure p^s . The material moduli β_1 and β_4 , are the material parameters that reflect the distribution of the granular particles, and β_0 plays the role akin to pressure in a compressible fluid and is given by an equation of state. The material modulus β_2 is again a viscosity akin to the second coefficient of viscosity in a compressible fluid, and β_3 denotes the viscosity of the

granular particles. Rajagopal and Massoudi [24] have outlined an experimental/theoretical program to determine these material moduli. We should remark here that other theories such as the kinetic theory of gases [38] or particle simulation [39,40], etc., can also be used to derive an expression for the stress tensor of the granular materials. For a recent and detailed review of all these issues, we refer the reader to the work of Hutter and Rajagopal [41].

In the majority of fluid–solid mixtures, the fluid is either a gas or water. Therefore, it seems appropriate to assume that the fluid behaves as a linearly viscous fluid (Newtonian), whose constitutive equation is:

$$\mathbf{T}^f = -p^f \mathbf{1} + \lambda_f (\text{tr } \mathbf{D}^f) \mathbf{1} + 2\mu_f \mathbf{D}^f. \quad (20)$$

In the above-equation, p^f is the fluid pressure, λ_f and μ_f are the viscosities. If the fluid is incompressible, then p^f is one of the unknown quantities in the problem that would have to be calculated. If the fluid is compressible, an equation of state is needed. In general, p^f , λ_f , and μ_f are functions of ρ_f .

The above constitutive relations, Eqs. (19) and (20), express the behavior of the granular materials and the fluid phase, respectively, in their reference states. However, since we are interested in the motion of the mixture in the current configuration, these equations somehow would have to be modified. Departing from previous approaches, we shall assume that the stress \mathbf{T}^s and \mathbf{T}^f in the pure state are weighted by ν and $(1 - \nu)$, respectively. Thus, we shall assume that:

$$\mathbf{T}_1 = \mathbf{T}^s, \quad (21)$$

and

$$\mathbf{T}_2 = (1 - \nu) \mathbf{T}^f, \quad (22)$$

and thus the total Cauchy stress in the mixture \mathbf{T} is given by:

$$\mathbf{T} = \mathbf{T}^s + (1 - \nu) \mathbf{T}^f. \quad (23)$$

When $\nu \rightarrow 0$, then $\mathbf{T} \rightarrow \mathbf{T}^f$, while when $\nu \rightarrow 1$, then $\mathbf{T} \rightarrow \mathbf{T}^s$ as we would expect. It is important to recognize that this limit, i.e., as $\nu \rightarrow 1$, is an ‘ideal’ limit in the sense that in reality there is a practical maximum packing beyond which the particles cannot be packed anymore; this is usually designated as ν_{\max} .

In the previous approaches, the weighing of the stresses of the components were done according to the ratio of their current density in the mixture to their pure density. Thus, for instance if ρ_s and ρ_f represented the pure density of each phase, and ρ_1 and ρ_2 represented the current density of each phase in the mixture, then \mathbf{T}^s was weighted by ρ_1/ρ_s and \mathbf{T}^f by ρ_2/ρ_f . However, the weighing could also be done with regard to the mass fraction of the constituents in the mixture rather than their mass fraction with regard to their respective pure state. Thus, we could weight the first constituents by ρ_1/ρ , and the second constituent by ρ_2/ρ . In this sense, the former is

a weighting in a Lagrangean manner with weighting done with regard to the reference mass fractions while the latter could be viewed as an Eulerian approach in which the weighting is done with respect to the ratio of the current mass fractions. Notice that in our modeling we have neglected the dependence of stresses on $\nabla \rho_1$ and $\nabla \rho_2$. Introducing higher gradients in the theory always introduces the attendant difficulty of specifying additional boundary conditions. Later, we shall see that even dependence of $\nabla \rho_1$ or $\nabla \nu$ introduces difficulties with regard to boundary condition.

In the above model, we shall assume that β_0 , β_2 , $\beta_4 \rightarrow 0$ as $\nu \rightarrow 0$, and thus $\mathbf{T}^s \rightarrow 0$ as $\nu \rightarrow 0$. This simply implies that when there are no particles, ($\nu = 0$), the stress in the granular materials, \mathbf{T}^s , would vanish. Also, notice that $\mathbf{T}_2 \rightarrow 0$ as $\nu \rightarrow 1$.

The mechanical interaction force between the mixture components, \mathbf{f} , is given as [42]:

$$\begin{aligned} \mathbf{f} = & A_1 \text{grad } \nu + A_2 F(\nu)(\mathbf{u}^s - \mathbf{u}^f) \\ & + A_3 \nu (2 \text{tr } \mathbf{D}_2^2)^{-1/4} \mathbf{D}_2(\mathbf{u}^s - \mathbf{u}^f) \\ & + A_4 \nu (\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{u}^s - \mathbf{u}^f) + A_5 \mathbf{a}_{\text{vm}}, \end{aligned} \quad (24)$$

where \mathbf{a}_{vm} is a properly frame-invariant measure of the relative acceleration between the mixture components and $F(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The terms in Eq. (24) reflect the presence of density gradients,¹ drag, ‘‘slip-shear’’ lift, ‘‘spin’’ lift, and virtual mass, respectively. Muller’s work [43] indicates that a term of the form $A_1 \text{grad } \nu$ must be included in the interaction to get consistently formulated problems. The term multiplying A_3 is a generalization of Saffman’s [44,45] single particle result first proposed in this form by McTigue et al. [46]. The material coefficients A_1 to A_5 need to be determined experimentally, or via alternative theories such as the kinetic theory. The general trend is to use the experimental correlations based on single particle measurements and, whenever applicable, by suitable methods these correlations are generalized for an assembly of particles. There are few experiments except in the case of drag, that give specific correlations for these coefficients. For a review of these issues, the reader is referred to the work of Johnson et al. [47]. Furthermore,

$$\mathbf{D}_1 = \frac{1}{2}(\mathbf{L}_1 + \mathbf{L}_1^T), \quad \mathbf{D}_2 = \frac{1}{2}(\mathbf{L}_2 + \mathbf{L}_2^T) \quad (25)$$

$$\mathbf{W}_1 = \frac{1}{2}(\mathbf{L}_1 - \mathbf{L}_1^T), \quad \mathbf{W}_2 = \frac{1}{2}(\mathbf{L}_2 - \mathbf{L}_2^T) \quad (26)$$

¹ The actual form of this interaction should include the terms $\alpha_1 \text{grad } \rho_1 + \alpha_2 \text{grad } \rho_2$ where α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to $A_1 \text{grad } \nu$ where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises from the granular solid stress tensor, this term will be neglected in the present work.

where

$$\mathbf{L}_1 = \text{grad } \mathbf{u}^s; \quad \mathbf{L}_2 = \text{grad } \mathbf{u}^f. \quad (27)$$

Substituting Eqs. (19), (20) and (8) into Eq. (12), we obtain the balance of linear momentum for the mixture:

$$\begin{aligned} \rho_s \nu \frac{d\mathbf{u}}{dt} + \rho_f(1-\nu) \frac{d\mathbf{v}}{dt} \\ = \text{div} \left[\{ \beta_0 + \beta_1 \nabla \nu \cdot \nabla \nu + \beta_2 \text{tr } \mathbf{D}^s \} \mathbf{1} \right. \\ \left. + \beta_3 \mathbf{D}^s + \beta_4 \nabla \nu \otimes \nabla \nu \right] + \text{div} \left(-(1-\nu) p^f \mathbf{1} \right. \\ \left. + \lambda_f(1-\nu) (\text{tr } \mathbf{D}^f) \mathbf{1} + 2\mu_f(1-\nu) \mathbf{D}^f \right) \\ \left. + \rho_s \nu \mathbf{b} + \rho_f(1-\nu) \mathbf{b}. \right. \end{aligned} \quad (28)$$

Taking the limit of this equation as $\nu \rightarrow 0$ (no particles), we should obtain the Navier–Stokes equation. This is only possible if [48]:

$$\beta_0 \rightarrow 0; \quad \beta_2 \rightarrow 0; \quad \beta_3 \rightarrow 0 \text{ as } \nu \rightarrow 0 \quad (29)$$

which was mentioned earlier. Therefore, Eq. (28), as $\nu \rightarrow 0$, becomes:

$$\rho_f \frac{d\mathbf{v}}{dt} = \text{div} \left[\{ -p^f \mathbf{1} + \lambda_f (\text{tr } \mathbf{D}^f) \mathbf{1} \} + 2\mu_f \mathbf{D}^f \right] + \rho^f \mathbf{b} \quad (30)$$

which is indeed the Navier–Stokes equation. Now, taking the limit of Eq. (28) as $\nu \rightarrow 1$ or $\nu \rightarrow \nu_{\max}$ (dense flow of granular materials), we obtain the equation of motion for flowing granular materials where the effects or the presence of the interstitial fluid is neglected. Therefore, Eq. (28), as $\nu \rightarrow 1$, becomes:

$$\begin{aligned} \rho_s \nu \frac{d\mathbf{u}}{dt} = \text{div} \left[\{ \beta_0 + \beta_1 \nabla \nu \cdot \nabla \nu + \beta_2 \text{tr } \mathbf{D}^s \} \mathbf{1} \right. \\ \left. + \beta_3 \mathbf{D}^s + \beta_4 \nabla \nu \otimes \nabla \nu \right] + \rho_s \nu \mathbf{b}. \end{aligned} \quad (31)$$

To obtain explicit dependence of β_i s on ν , we use a Taylor series expansion. Thus:

$$\beta_i(\nu) = \beta_{i0} + \beta_i(0) \nu + O(|\nu^2|); \quad i = 0, 1, 2, 3, 4 \quad (32)$$

where $O(|\nu^2|)$ indicates terms of higher order than ν . Eq. (29) implies that $\beta_{i0} = 0$ for $i = 0, 2$, and 3 . Therefore, as a first approximation, the material properties β_0 through β_4 must have the following forms:

$$\beta_i = k_i \nu; \quad i = 0, 2, 3 \quad (33)$$

$$\beta_j = k_{0j} \nu + k_j \nu; \quad j = 1, 4. \quad (34)$$

Through a simple analysis, Rajagopal and Massoudi [24] have shown that if the constitutive Eq. (19) is to apply at equilibrium (static) case, then $k_0 < 0$ and therefore:

$$\beta_0 = -\kappa \nu. \quad (35)$$

With such structure, this model is capable of predicting one of the normal stress differences, which in the context of granular materials, is often referred to as ‘dilatancy’. This phenomenon was first observed by Reynolds [49] and

explained later by Bagnold [50]. To better understand the significance of β_i s, Gudhe et al. [51] used this model to study the flow and heat transfer of granular materials flowing down an inclined plane.

With these restrictions, the stress tensor of the mixture, Eq. (23), becomes:

$$\begin{aligned} \mathbf{T} = \{ -\kappa \nu - (1-\nu) p^f + (k_{01} + k_1 \nu) \nabla \nu \cdot \nabla \nu \\ + (k_2 \nu) \text{tr } \mathbf{D}^s + \lambda_f(1-\nu) \text{tr } \mathbf{D}^f \} \mathbf{1} + k_3 \nu \mathbf{D}^s \\ + (k_{04} + k_4 \nu) \nabla \nu \otimes \nabla \nu + 2\mu_f(1-\nu) \mathbf{D}^f. \end{aligned} \quad (36)$$

In Section 2.4, we will discuss the implications of modeling a mixture of fluid–solid particles as being composed of granular materials and a linearly viscous fluid.

2.4. On the viscosity of the mixture

Historically, the problem of the theoretical determination of the viscosity of an infinitely dilute suspension consisting of an incompressible Newtonian fluid and rigid spherical particles was studied by Einstein [52]. He derived the classical formula for the effective viscosity of the suspension:

$$\mu^* = \mu_f(1 + 2.5\nu), \quad (37)$$

where μ_f is the viscosity coefficient of the fluid and ν is the volume fraction of spheres in the suspension. Einstein assumed that ν is very small compared with unity. This indicates that the particles are far apart from each other and are not interacting with each other. Taylor [53] showed that if the spheres are small drops of another fluid, then the viscosity of the suspension is given by:

$$\mu^* = \mu_f \left[1 + 2.5\nu \left(\frac{\mu' + \frac{2}{5}\mu_f}{\mu' + \mu_f} \right) \right], \quad (38)$$

where μ' is the viscosity of the liquid drops and μ_f is the viscosity of the host fluid. Later, Batchelor and Green [54] derived a formula for the effective viscosity including the terms of order ν^2 . They showed that:

$$\mu^* = \mu_f(1 + 2.5\nu + 7.6\nu^2), \quad (39)$$

where again $\nu \ll 1$. There have been many other attempts, both experimentally and analytically, to derive an equation for the viscosity of the mixture (or suspension); for a review of transport properties of two-phase materials, the reader is referred to the work of Batchelor [55].

Sampaio and Williams [56] used a modified version of mixture theory to obtain an expression for the effective viscosity of a mixture of two liquids. They showed that if the velocities of the two liquids are equal, then the effective viscosity of the mixture is given by:

$$\mu = \mu_A + \mu_B + \mu_{AB}, \quad (40)$$

where μ_A and μ_B are the viscosities of the two fluids and μ_{AB} is defined as the ‘mutual viscosity’, which arises

due to the interactions of the molecules of liquid A and liquid B, and furthermore, all the viscosities are assumed to be functions of the volume fraction. If this is the case, then it is not clear how μ_{AB} could exist [cf. Eq. (40)] if the two liquids have the same velocity. They also compared Eq. (40) with the expression given by Dolezalek and Schulze (see reference in Ref. [56]), where:

$$\mu = \phi^2 \eta_A + 2\phi(1 - \phi)\eta_{AB} + (1 - \phi)^2 \eta_B, \quad (41)$$

where η_A and η_B are the pure (unmixed) viscosities of the two liquids. Sampaio and Williams [56] concluded that:

$$\mu_A = \phi^2 \eta_A; \quad \mu_B = (1 - \phi)^2 \eta_A; \quad \mu_{AB} = 2\phi(1 - \phi)\eta_{AB}. \quad (42)$$

Nunziato [57], using mixture theory, defined the viscosity of a mixture of a fluid and particles as:

$$\mu = (1 - \nu)\mu_f + \nu\mu_s. \quad (43)$$

Again, for the case of no slip between the two phases (i.e., equal velocities) Nunziato [57] suggested that the solid viscosity, in the low concentration limit [cf. Eq. (37)] can be estimated as:

$$\mu_s = 3.5\mu_f. \quad (44)$$

If the mixture of fluid–solids is assumed to be composed of a saturated granular media infused with a linearly viscous fluid, then the stress tensor for the mixture according to the present formulation is given in Eq. (36). It can be seen that it is not an easy matter to define a mixture viscosity by simply looking at Eq. (36). That is, if somehow we can rewrite this equation as:

$$\mathbf{T} = -p_m \mathbf{1} + 2\mu_m \mathbf{D}_m + \alpha \nabla \nu \otimes \nabla \nu, \quad (45)$$

where p_m can be thought of as a mixture pressure and \mathbf{D}_m as the stretching rate for the mixture, then we can consider the meaning of μ_m (the mixture viscosity) and how it can possibly be measured. This, however, is not an easy task. Instead, if we consider the ‘ideal’ case of equal velocities ($\mathbf{u} = \mathbf{v}$) (i.e., the particles are simply being carried by the host fluid), then Eq. (36) is rewritten as:

$$\begin{aligned} \mathbf{T} = & \{-\kappa\nu - (1 - \nu)p^f + (k_{01} + k_1\nu)\nabla\nu \cdot \nabla\nu \\ & + (k_2\nu)\text{tr}\mathbf{D} + \lambda_f(1 - \nu)\text{tr}\mathbf{D}\}\mathbf{1} \\ & + [k_3\nu + 2\mu_f(1 - \nu)]\mathbf{D} + (k_{04} + k_4\nu)\nabla\nu \otimes \nabla\nu. \end{aligned} \quad (46)$$

This indicates that the flow characteristics of a suspension can be described, in this special case, by a constitutive relation similar to that of a Korteweg-type fluid [32]. (The previous approaches have indicated that the suspension behaves like a Newtonian fluid where the effective viscosity is a function of the volume fraction.) Therefore, if one performs a simple shear flow experiment, the term multiplying \mathbf{D} in the above equation can be thought of as the viscosity of the mixture, i.e.:

$$\mu_m = \frac{k_3}{2}\nu + \mu_f(1 - \nu). \quad (47)$$

If at the same time, the volume fraction of the solid particles and the viscosity of the fluid are independently measured, we can obtain the viscosity of solid particles k_3 . Eq. (47) simply indicates a relationship between the viscosity of the mixture and the viscosities of the two constituents provided:

- (i) The fluid is modeled as a linearly viscous fluid, represented by Eq. (20);
- (ii) The solid particles are modeled as a granular media represented by a constitutive relation, Eq. (19);
- (iii) The total stress of the mixture is weighted according to Eq. (23);
- (iv) The material moduli of the granular media are approximated as linear functions, Eqs. (33) and (34), with the restriction mentioned; and
- (v) The fluid and solid velocities are the same.

In the more complicated case of \mathbf{u} not being equal to \mathbf{v} , there is no clear analogy of a mixture viscosity to the viscosity of a single fluid.

2.5. Flow through a pipe

Consider the pressure-driven flow of a mixture through an infinitely long pipe of circular cross-section. Let z denote the direction of the flow and let the wall of the pipe be located at $r = 1$. If the flow is steady and laminar, the velocity profiles and solids distribution can be assumed to have the form: (for simplicity we replace \mathbf{u}^f and \mathbf{u}^s by \mathbf{V}_1 and \mathbf{V}_2 , respectively).

$$\mathbf{V}_1 = V(r)\mathbf{e}_z; \quad \mathbf{V}_2 = U(r)\mathbf{e}_z; \quad \nu = \nu(r). \quad (48)$$

The balance of mass equations are automatically satisfied by these assumptions. Virtual mass effects are not present in this steady unidirectional flow; body forces are also neglected. The detailed derivation of the following equations are given in the work of Johnson et al. [58] and we will only mention them here for the sake of completeness. Substituting Eqs. (48) and (24) into Eqs. (6) and (7) and using Eqs. (19), (20), (33) and (34) we have:

$$\begin{aligned} (1 - \nu) \left(V'' + \frac{1}{r} V' \right) - \nu' V' - Re(1 - \nu) \frac{\partial P}{\partial z} \\ + C_2 Re \mathcal{F}(\nu)(U - V) = 0 \end{aligned} \quad (49)$$

$$\begin{aligned} \nu' P - (1 - \nu) \frac{\partial P}{\partial r} + \frac{1}{2} C_3 \nu |V'|^{-\frac{1}{2}} V' (U - V) \\ + \frac{1}{2} C_4 \nu (V' - U') (U - V) = 0 \end{aligned} \quad (50)$$

$$\begin{aligned} B_3 \left[(\nu + \nu^2) \left(U'' + \frac{1}{r} U' \right) + (2\nu + 1) \nu' U' \right] \\ - 2C_2 \mathcal{F}(\nu)(U - V) = 0 \end{aligned} \quad (51)$$

$$\begin{aligned} B_0 \nu' + (B_1 + B_4) [2(1 + \nu + \nu^2) \nu' \nu'' + (2\nu + 1)(\nu')^3] \\ + B_4 \frac{1}{r} (\nu')^2 (1 + \nu + \nu^2) - \frac{1}{2} C_3 \nu |V'|^{-\frac{1}{2}} V' (U - V) \\ - \frac{1}{2} C_4 \nu (V' - U') (U - V) = 0 \end{aligned} \quad (52)$$

where the prime denotes a derivative with respect to r . The dimensionless numbers are:

$$Re = \frac{\rho_0 u_0 L}{\mu_f}; \quad \Lambda = \frac{\lambda_f}{\rho_0 u_0 L}; \quad Fr = \frac{u_0^2}{Lg} \quad (53)$$

$$B_0^* = \frac{\beta_0}{\rho_0 u_0^2}; \quad B_1^* = \frac{\beta_1}{\rho_0 u_0^2 L^2}; \quad B_2^* = \frac{\beta_2}{\rho_0 u_0 L} \quad (54)$$

$$B_3^* = \frac{\beta_3}{\rho_0 u_0 L}; \quad B_4^* = \frac{\beta_4}{\rho_0 u_0^2 L^2}; \quad C_1 = \frac{A_1}{\rho_0 u_0^2} \quad (55)$$

$$C_2 = \frac{A_2 L}{\rho_0 u_0}; \quad C_3 = \frac{A_3 L^{1/2}}{\rho_0 u_0^{1/2}}; \quad C_4 = \frac{A_4}{\rho_0}; \quad C_5 = \frac{A_5}{\rho_0}. \quad (56)$$

Eliminating the pressure term from Eqs. (49) and (50) by cross-differentiating yields:

$$\begin{aligned} (1 - \nu) V''' + \left[\frac{1}{r} (1 - \nu) - 2\nu' \right] V'' \\ + \left[-\frac{1}{r^2} (1 - \nu) - \nu'' - \frac{1}{r} \nu' \right] V' \\ + C_2 Re [\mathcal{F}'(U - V) + \mathcal{F}(U' - V')] = 0. \end{aligned} \quad (57)$$

Now let us consider Eqs. (51) and (52) in the limiting case as $\nu \rightarrow 0$. Note that it is assumed, based on physical considerations, that $\mathcal{F}(\nu) \rightarrow 0$ as $\nu \rightarrow 0$. If $\nu \equiv 0$, Eqs. (51) and (52) vanish, and Eq. (57) becomes:

$$\left[\frac{1}{r} (rV')' \right]' = 0 \quad (58)$$

which is consistent with the Poiseuille flow of a pure fluid, with the solution being $V = c/4(r^2 - R^2)$ where c is a constant.

Eqs. (51), (52) and (57) must be solved numerically. To do so the following definitions are introduced:

$$\pi_1 = C_2 Re; \quad \pi_2 = \frac{2C_2}{B_3}; \quad \pi_3 = \frac{B_0}{B_1 + B_4} \quad (59)$$

$$\pi_4 = \frac{B_4}{B_1 + B_4}; \quad \pi_5 = \frac{-C_3}{2(B_1 + B_4)}; \quad \pi_6 = \frac{-C_4}{2(B_1 + B_4)}. \quad (60)$$

These dimensionless parameters have physical interpretations. π_1 is the ratio of the drag force exerted on the fluid by the solid to the viscous forces within the fluid phase. π_2 is the ratio of the drag force exerted on the solid by the fluid to the viscous forces within the solid component. π_3 and π_4 are the combinations of the material coefficients from the granular solid constitutive equation; they affect the distribution of solid in the flow but do not directly influence the velocity profiles of either fluid or solid. π_5 incorporates the effect of slip-shear lift into the equations; π_6 is the spin lift coefficient. The fluid and solid velocity fields exert an effect on the distribution of

the granular solid only when $\pi_5 \neq 0$ and/or $\pi_6 \neq 0$. With these definitions Eqs. (51), (52) and (57) are rewritten as:

$$\begin{aligned} (1 - \nu) V''' + \left[\frac{1}{r} (1 - \nu) - 2\nu' \right] V'' \\ + \left[-\frac{1}{r^2} (1 - \nu) - \nu'' - \frac{1}{r} \nu' \right] V' \\ + \pi_1 [\mathcal{F}'(U - V) + \mathcal{F}(U' - V')] = 0 \end{aligned} \quad (61)$$

$$\begin{aligned} (\nu + \nu^2) \left(U'' + \frac{1}{r} U' \right) + (2\nu + 1) \nu' U' \\ - \pi_2 \mathcal{F}(\nu)(U - V) = 0 \end{aligned} \quad (62)$$

$$\begin{aligned} 2(1 + \nu + \nu^2) \nu'' \nu' + (2\nu + 1)(\nu')^3 + \pi_3 \nu' \\ + \pi_4 \frac{1}{r} (\nu')^2 (1 + \nu + \nu^2) + \pi_5 \nu |V'|^{-\frac{1}{2}} V' (U - V) \\ + \pi_6 \nu (V' - U') (U - V) = 0. \end{aligned} \quad (63)$$

Notice that when $\pi_5 = \pi_6 = 0$, Eq. (63) is not coupled to Eqs. (61) and (62) through the velocity field, and thus Eq. (63) can be solved independently to determine ν . Based on Batchelor's [59] result for sedimentation of a dilute suspension of spheres, Drew [33] proposes the following approximation for $\mathcal{F}(\nu)$.

$$\mathcal{F}(\nu) = \nu(1 + 6.55\nu). \quad (64)$$

This expression is used here with the understanding that it is not valid as $\phi \rightarrow 0$ (or as $\nu \rightarrow \nu_m$).

2.6. Boundary conditions

Adherence (or the no-slip) boundary conditions are imposed on both the constituents at the pipe wall:

$$U(1) = V(1) = 0. \quad (65)$$

The flow rate of the mixture is also prescribed as a condition. The mass flow rate of a two component mixture is defined as:

$$Q_m = 2\pi \int_0^1 \rho_m V_m r dr = 2\pi \int_0^1 [(1 - \nu) \rho_f V + \nu \rho_s U] r dr. \quad (66)$$

A volumetric flow rate may be defined as:

$$Q = 2\pi \int_0^1 V_m r dr = 2\pi \int_0^1 [(1 - \nu) V + \nu U] r dr. \quad (67)$$

When the mixture is neutrally buoyant, i.e., $\rho_f = \rho_s = \rho$, these two quantities are simply related by $Q_m = \rho Q$. The volumetric flow rate of the mixture is the quantity prescribed in the present work.

The appropriate conditions on ν are the boundary value of ν at the pipe wall and a prescribed average volume fraction, defined through:

$$N = 2\pi \int_0^1 \nu r dr. \quad (68)$$

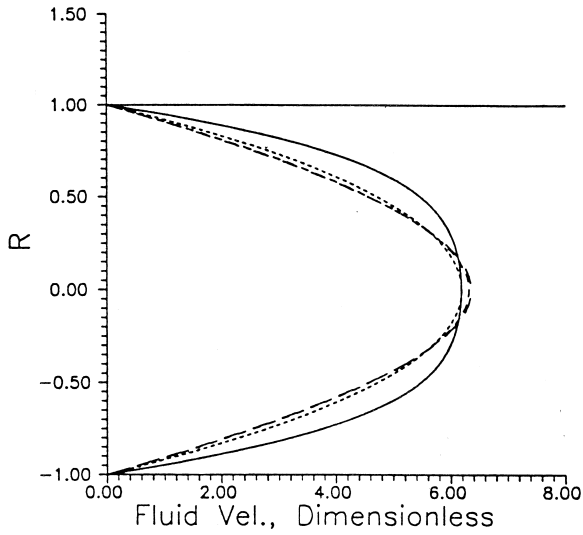


Fig. 1. Effect of π_2 on fluid velocity. II; (—: $\pi_2 = 1$; —: $\pi_2 = 10$; ---: $\pi_2 = 100$; — — —: $\pi_2 = 1000$).

Numerical values for ν on the boundary could be obtained from experiments such as those performed by Segre and Silberberg [60]. Kadambi et al. [61] observe experimentally that $\nu \rightarrow 0$ as $r \rightarrow 1$ for small values of N .

2.7. Numerical results

The full treatment of this problem is given in the work of Johnson et al. [42,58]. In this section, we only discuss the significance of the dimensionless numbers and their influence on the flow field. Issues such as pressure drop and skin friction which were not considered before are given more consideration here. With regard to the velocity and concentration profiles, only a few representative figures will be presented.

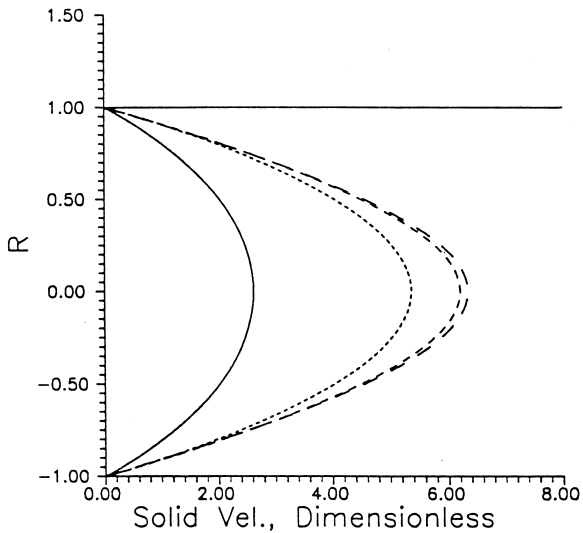


Fig. 2. Effect of π_2 on solid velocity. II; (—: $\pi_2 = 1$; —: $\pi_2 = 10$; ---: $\pi_2 = 100$; — — —: $\pi_2 = 1000$).

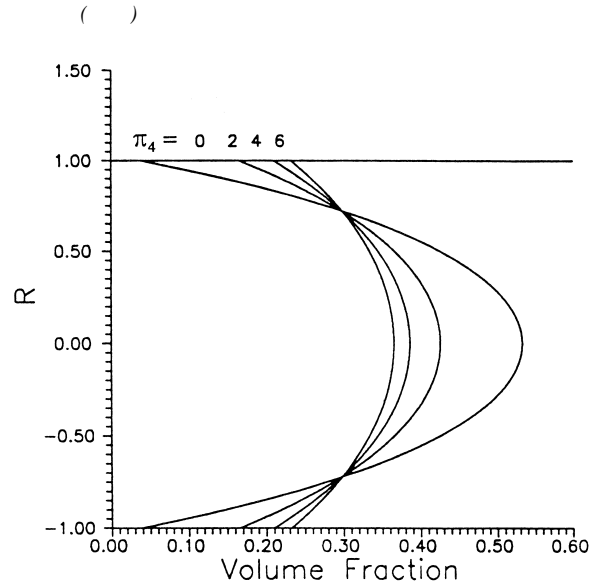


Fig. 3. Flow in a pipe: effect of π_4 ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_3 = 3$; $\pi_5 = \pi_6 = 0$; $N = 0$; $Q = 3$).

There are some restrictions on the values the parameters may take. Clearly, the drag coefficient, Reynolds number, and the material viscosities must be greater than zero and therefore, π_1 and π_2 must be also greater than zero. Similarly, the lift coefficients π_5 and π_6 must be less than zero for the direction of lift to match experimental observations [62]. Based on the results obtained for flow between parallel plates [47], we assume $B_1 + B_4 < 0$. Since B_0 is also less than zero [24], π_3 is greater than zero and π_4 is also assumed to be greater than zero.

Increasing values of π_1 result in a decrease in the maximum fluid velocity and a “flattening” of the profile. Note that, as π_1 gets large there are local maxima in the fluid velocity profiles. Though it does not change shape, the solid velocity also decreases due to the coupling of the

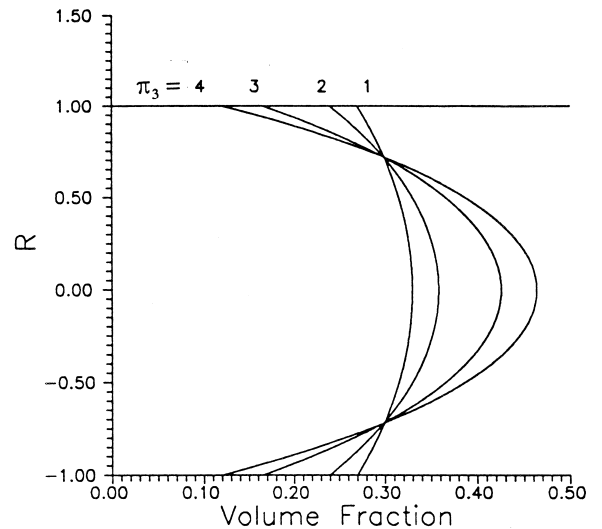


Fig. 4. Flow in a pipe: effect of π_3 ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_4 = 4$; $\pi_5 = \pi_6 = 0$; $N = 0$; $Q = 3$).

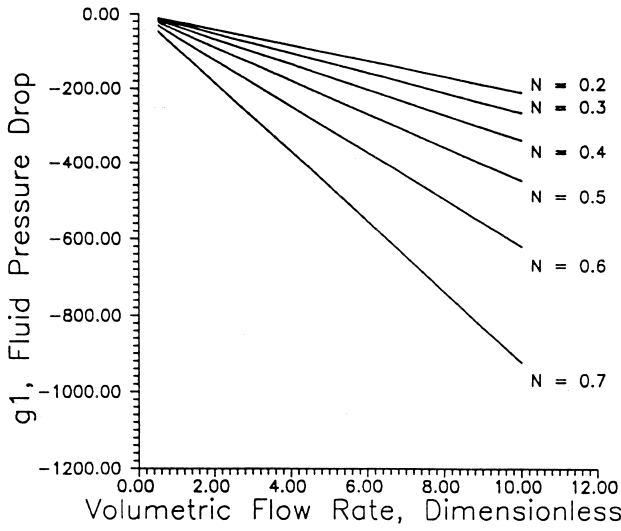


Fig. 5. Flow in a pipe: effect of N ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_3 = 4$; $\pi_4 = 4$; $\pi_5 = \pi_6 = 0$).

two equations. Physically, an increase in the value of π_1 corresponds to either an increase in the drag coefficient or an increase in the Reynolds number or both, and thus the results obtained are in keeping with physics.

Increasing π_2 results in a decrease in fluid velocity and a corresponding increase in solid velocity (Figs. 1 and 2). π_2 does not affect the shape of either the fluid or solid velocity profile. Physically, an increase in the value of π_2 corresponds to either an increase in the drag coefficient or a decrease in the dimensionless number B_3 . The drag of the fluid on the solid is the only force driving the motion of the granular material in this problem. Thus, if $\pi_2 = 0$, U is identically zero; if $\pi_2 > 100$ the $U \approx V$ everywhere.

Figs. 3 and 4 show plots of volume fraction vs. r for various values of π_3 and π_4 (with $\pi_5 = \pi_6 = 0$). Note

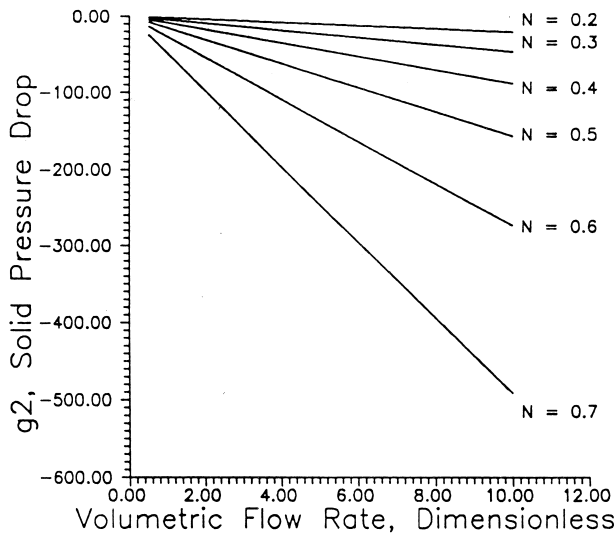


Fig. 6. Flow in a pipe: effect of N ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_3 = 4$; $\pi_4 = 4$; $\pi_5 = \pi_6 = 0$).

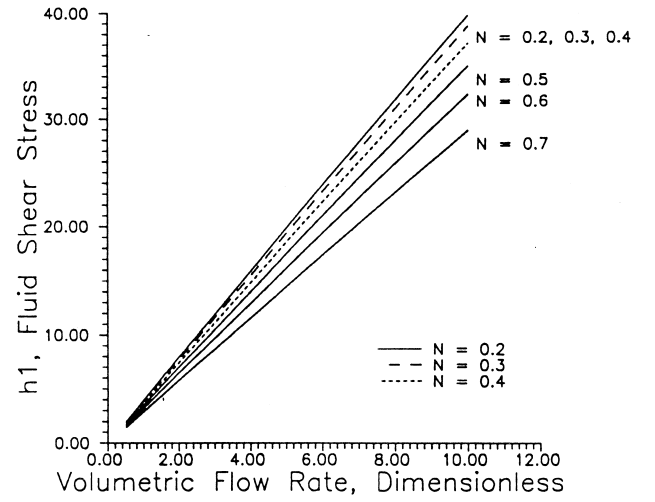


Fig. 7. Flow in a pipe: effect of N ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_3 = 4$; $\pi_4 = 4$; $\pi_5 = \pi_6 = 0$).

that an integral condition is used and the area under each of the curves are the same, although the boundary condition at the tube wall is different for each curve. Recall when $\pi_5 = \pi_6 = 0$, the equation for ν is not coupled to either the fluid or solid velocity field. The distribution of the granular solid is therefore not influenced by π_1 or π_2 . The velocity profiles, however, are strongly influenced by the choice of π_3 and π_4 . With regard to flowing granular materials, some interesting experimental observations are provided in the work of Ahn et al. [63].

2.8. Applications

Once ν , U , and V are known, several quantities of interest may be derived from the solutions. The pressure gradient may be calculated by simply rearranging Eq. (49):

$$Re \left(\frac{\partial P}{\partial z} \right) = V'' + \frac{1}{r} V' - \left(\frac{1}{1 - \nu} \right) [\nu' V' - C_2 Re \mathcal{F}(\nu)(U - V)] \quad (69)$$

or, combining this equation with Eq. (52):

$$\frac{\partial P}{\partial z} = \left(\frac{B_3}{2} \right) \left[\frac{(\nu + \nu^2) \left(U'' + \frac{1}{r} U' \right) + (2\nu + 1) \nu' U'}{1 - \nu} \right] + \frac{1}{Re} \left[\frac{(1 - \nu) \left(V'' + \frac{1}{r} V' \right) - \nu' V'}{1 - \nu} \right]. \quad (70)$$

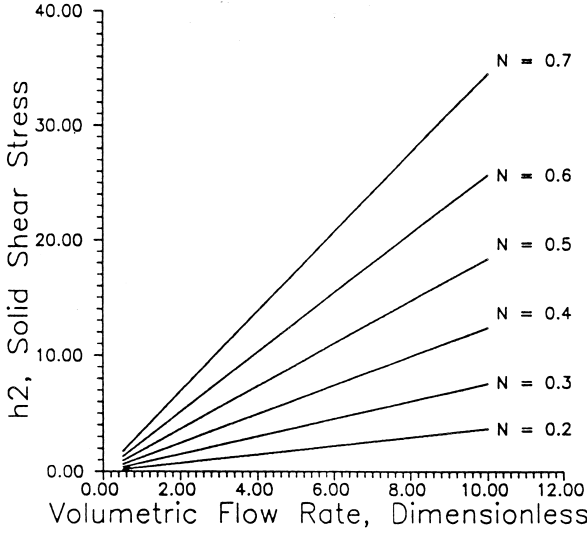


Fig. 8. Flow in a pipe: effect of N ; ($\pi_1 = 10$; $\pi_2 = 5$; $\pi_3 = 4$; $\pi_4 = 4$; $\pi_5 = \pi_6 = 0$).

This expression may be rewritten as:

$$\frac{\partial P}{\partial z} = g_1 \left(\frac{1}{Re} \right) + g_2 \left(\frac{B_3}{2} \right) \quad (71)$$

where

$$g_1 = \frac{(1 - \nu) \left(V'' + \frac{1}{r} V' \right) - \nu' V'}{1 - \nu} \quad (72)$$

$$g_2 = \frac{(\nu + \nu^2) \left(U'' + \frac{1}{r} U' \right) + (2\nu + 1) \nu' U'}{1 - \nu}. \quad (73)$$

Here g_1 represents the pressure drop due to the fluid component of the mixture and g_2 represents the pressure drop due to the solid component. The average values of the function g_1 and g_2 are plotted (vs. volumetric flow rate for several values of N) in Figs. 5 and 6 for a representative set of dimensionless parameter values.

In addition to the pressure drop, the skin friction at the wall is also of practical interest. This quantity is related to the traction vector (or force) at the wall which is:

$$\mathbf{t} = \mathbf{T}_m \cdot \mathbf{n} = \begin{pmatrix} P(1 - \nu) - B_0 \nu - (B_1 + B_4)(1 + \nu + \nu^2)(\nu')^2 \\ 0 \\ -\frac{1}{Re}(1 - \nu)V' - \frac{1}{2}B_3(\nu + \nu^2)U' \end{pmatrix}. \quad (74)$$

The shear stress at the wall can be written:

$$\sigma = h_1 \left(\frac{1}{Re} \right) + h_2 \left(\frac{B_3}{2} \right) \quad (75)$$

where

$$h_1 = -(1 - \nu)V'; \quad h_2 = -(\nu + \nu^2)U'. \quad (76)$$

Here h_1 represents the shear due to the fluid component of the mixture and h_2 represents the shear due to the solid component. The functions h_1 and h_2 are plotted (vs. volumetric flow rate for several values of N) in Figs. 7 and 8 for a representative set of dimensionless parameter values.

3. Nomenclatures

\mathbf{a}	acceleration vector
a_{vm}	frame-indifferent relative acceleration
A_i	interaction coefficient, $i = 1$ to 5
\mathbf{b}	body force vector
B_i^*	dimensionless β s
B_i	constant part of B_i^*
π_i	dimensionless parameters; $i = 1$ to 6
C_i	dimensionless A s
\mathbf{D}	stretching tensor
\mathbf{f}	interaction force vector
Fr	Froude number
F	volume fraction dependence of drag
g	gravity
g_1	dimensionless pressure drop, fluid
g_2	dimensionless pressure drop, solid
h_1	dimensionless shear stress, fluid
h_2	dimensionless shear stress, solid
\mathbf{I}	identity tensor
\mathbf{L}	gradient of a velocity vector
N	average volume fraction
p	fluid pressure
P	dimensionless fluid pressure
Q	volumetric flow rate of mixture
Q_m	mass flow rate of mixture
r	dimensionless radial coordinate
Re	Reynolds number
\mathbf{T}	stress tensor
\mathbf{U}	solid velocity
\mathbf{v}	velocity vector
V	fluid velocity
\mathbf{V}	dimensionless velocity vector
\mathbf{W}	spin tensor
\mathbf{x}	position vector
\mathbf{X}	dimensionless position vector
z	direction of flow
β_i	granular solid coefficients, $i = 0$ to 4
δ	Dirac delta function
λ_f	second coefficient of viscosity, fluid
Λ	dimensionless λ_f
μ	first coefficient of fluid viscosity
ν	volume fraction of the solid particles
π	dimensionless parameter
ρ	density
σ	dimensionless shear stress
τ	dimensionless time
ϕ	volume fraction of fluid

Subscripts

2,f	referring to the fluid phase
1,s	referring to the solid phase
m	referring to mixture

Superscripts

T	transpose
*	dimensionless quantity

Other symbols

div	divergence operator
grad	gradient operator
tr	trace of a tensor
∇	gradient symbol
\otimes	Outer (dyadic) product
\cdot	Inner (Scalar) product

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